SHEAF QUANTIZATION OF GEOMETRICALLY BOUNDED EXACT LAGRANGIANS

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Abstract. We construct sheaf quantizations for any exact Lagrangian submanifold in a cotangent bundle with proper primitive and tubular neighbourhood of positive radius with respect to any complete conic Riemannian metric on the cotangent bundle.

1. Main results

Consider a Liouville manifold. Closed exact Lagrangian submanifolds, and more generally exact Lagrangian submanifolds with ideal Legendrian boundary, define objects in the Fukaya category. In microlocal sheaf theory, such results are also known due to Guillermou [3, 4] (and generalized by Jin–Treumann [7]):

Theorem 1.1 (Guillermou [3, 4], Jin–Treumann [7]). *Let $L \subset T^*M$ be an exact Lagrangian with ideal Legendrian boundary. Then there is a proper Legendrian lift $\widetilde{L} \subset J^1M$ (after a small Hamiltonian isotopy), and there exists a fully faithful functor*

$$\mu Sh_L(L) \hookrightarrow Sh_{\widetilde{L}}(M \times \mathbb{R})_0$$

that is left adjoint to the microlocalization functor, where the subscript 0 on the right hand side means 0 stalk at $-\infty$.

For closed Lagrangians (or Lagrangians with ideal Legendrian boundary) $L \subset X$, Weinstein neighbourhood theorem asserts that there is a tubular neighbourhood of $L \subset X$ symplectomorphic to a neighbourhood of the zero section $L \subset T^*L$. For noncompact Lagrangian submanifolds, the neighbourhood theorem could be nontrivial. To deal with this issue, we introduce the notion of an adapted metric following [1, Section 2.2.2]:

Definition 1.1. *A Riemannian metric g on a symplectic manifold X is adapted to the symplectic form ω on X if for any $H \in C^\infty(X)$,*

$$\|dH\|_g = \|X_H\|_g,$$

or equivalently $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ for some g-orthonormal coframing

$$\langle dx_1, \dots, dx_n, dy_1, \dots, dy_n \rangle .$$

Let $L \subset X$ be a submanifold. A (tubular) neighbourhood U of L of positive radius $r > 0$ with respect to a metric g on X is a (tubular) neighbourhood U such that for any $x \in X$ with $d_g(x, L) \leq r$, we have $x \in U$.

For Lagrangian cobordisms between closed Legendrians, such a tubular neighbourhood does not exist, for the simple reason that the symplectic area near the concave end of the symplectization has an upper bound, while a tubular neighbourhood of positive radius for the conical/cylindrical submanifold cannot have a bounded symplectic area.

In this short note, we explain how the exact same argument of Guillermou also constructs sheaf quantizations for exact Lagrangians with proper primitive and tubular neighbourhood of positive radius with respect to a given complete adapted metric:

Theorem 1.2. *Let $L \subset T^*M$ be an exact Lagrangian with a proper primitive bounded below on any compact subset $K \subset M$ and tubular neighbourhood of positive radius with respect to a given complete adapted metric in T^*M. Let $\widetilde{L} \subset J^1M$ be the proper Legendrian lift. Then there exists a fully faithful functor*

$$\mu Sh_L(L) \hookrightarrow Sh_{\widetilde{L}}(M \times \mathbb{R})_0$$

that is left adjoint to the microlocalization functor, where the subscript 0 on the right hand side means 0 stalk at $-\infty$.

We also have the following more general theorem, by essentially repeating the proof of the above main theorem line by line and replace all the categories $Sh(M \times \mathbb{R})$ by the localization category $Sh(M \times \mathbb{R})/Sh_{\leq 0}(M \times \mathbb{R})$. (The localization is necessary, for instance, for $\Gamma_{df} \subset T^*\mathbb{R}$ with $f = \tan x, x \in (-\pi/2, \pi/2)$.)

Theorem 1.3. *Let $L \subset T^*M$ be an exact Lagrangian with a tubular neighbourhood of positive radius with respect to a given complete adapted metric in T^*M. Let $\widetilde{L} \subset J^1M$ be the (possibly non-proper) Legendrian lift. Then there exists a fully faithful functor*

$$\mu Sh_L(L) \hookrightarrow Sh_{\widetilde{L}}(M \times \mathbb{R})/Sh_{\leq 0}(M \times \mathbb{R}),$$

that is left adjoint to the microlocalization functor.

Note that if we drop the assumption of the existence of tubular neighbourhood of positive radius, the sheaf quantization result will fail. See the examples of conic Lagrangian cobordisms between Legendrians studied in [11, Section 3].

2. Proof of the main theorem

2.1. **Doubling Functor for Noncompact Legendrians.** The doubling construction goes back to Guillermou [3, 4] (generalized by Jin–Treumann [7] and [9]), who defined an unconditional doubling functor for a compact exact Lagrangian

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)}(M \times \mathbb{R}),$$

and is also formulated in a different way by Nadler–Shende [12]. This construction plays a crucial role in the proof of the sheaf quantization theorems.

The goal of this section is to generalize the doubling construction to certain noncompact Legendrian submanifolds. Our main theorem is the following.

Definition 2.1. *A Riemannian metric g on a symplectization $X \times \mathbb{R}$ is adapted to the contact form $dt - \lambda$ on $X \times \mathbb{R}$ if for any $H \in C^\infty(X)$,*

$$|H| + \|dH\|_g = \|X_H\|_g,$$

or equivalently $dt - \lambda = dt - \sum_{i=1}^n y_i dx_i$ for some g-orthonormal coframing

$$\langle dx_1, \dots, dx_n, dy_1, \dots, dy_n, dt \rangle .$$

Theorem 2.1. *Let $\Lambda \subset J^1M$ be any proper smooth Legendrian. If there exists a complete adapted metric on J^1M such that Λ admits a tubular neighbourhood of positive radius $\epsilon_0 > 0$, then for any $0 < \epsilon < \epsilon_0$, there exists a fully faithful functor*

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)}(M \times \mathbb{R})$$

such that there is an exact triangle $T_{-\epsilon} \to T_\epsilon \to w_\Lambda \circ m_\Lambda$.

Moreover, we get an adjunction property for the doubling functor.

Theorem 2.2. *Let $\Lambda \subset J^1M$ be any proper smooth Legendrian. If there exists a complete adapted metric on J^1M such that Λ admits a tubular neighbourhood of positive radius $\epsilon_0 > 0$, then the doubling functor can be identified with the left adjoint of microlocalization*

$$m_\Lambda^* = \iota_\Lambda^* \circ w_\Lambda[-1] : \mu Sh_\Lambda(\Lambda) \to Sh_\Lambda(M \times \mathbb{R}),$$

where ι_Λ^* is the left adjoint to the inclusion $\iota_\Lambda : Sh_\Lambda(M \times \mathbb{R}) \hookrightarrow Sh(M \times \mathbb{R})$.

We remark that the assumption that Λ admits a tubular neighbourhood of positive radius is crucial, and indeed this is why there is not in general a doubling functor for conical Legendrian cobordisms with a uniform $\epsilon > 0$ (and why we need extra data near the negative end of the cobordisms).

We briefly recall the construction. First of all, given any $\mathscr{F} \in \mu Sh_\Lambda(\Lambda)$, for a sufficiently small open subsets $V \subset U \subset M \times \mathbb{R}$, the refined microlocal cut-off lemma [3, Lemma 6.7], [4, Lemma 10.2.5] or [9, Corollary 4.16] ensures that there exists a sheaf $\mathscr{F}_U \in Sh_\Lambda(U)$. Let

$$\Lambda_{\pm T} = \{(x,t,u;\xi,\tau,\nu) \mid \exists\, u > 0, (x,t;\xi,\tau) \in T_{\pm u}(\Lambda), \nu = -H_u \circ T_u(x,t;\xi,\tau)\}$$

be the Legendrian movies of Λ under the positive/negative Reeb flows. Let

$$T_\pm : Sh_\Lambda(M \times \mathbb{R}) \to Sh_{\Lambda_{\pm T}}(M \times \mathbb{R} \times (0,+\infty))$$

be the equivalences of sheaf categories induced by the corresponding Hamiltonian isotopies $T_\pm$ by Guillermou-Kashiwara-Schapira [5].

For $\epsilon > 0$ a sufficiently small number depending on U, V and Λ, we define

$$\widetilde{w}_\Lambda(\mathscr{F})_{V\times(0,\epsilon)} = \mathrm{Cone}(T_-(j_{U*}\mathscr{F}_U) \to T_+(j_{U*}\mathscr{F}_U))|_{V\times(0,\epsilon)}.$$

Locally, it follows from the Sato-Sabloff exact triangle of Ike [6, Section 4.3] or [10, Section 3.2], [9, Section 4.1] that the assignment is fully faithful. Then using the local-to-global argument, we have the well-definedness along with full faithfulness [9, Section 4.4]. In summary, we have the following proposition.

Proposition 2.3. *Let $\Lambda \subset J^1M$ be a proper subanalytic Legendrian subset. Then there exists a locally finite open covering $\{U_\alpha\}_{\alpha\in I}$ and an open refinement $\{V_\alpha\}_{\alpha\in I}$, with a collection of $\epsilon_\alpha > 0$ depending on U_α and V_α, such that there is a fully faithful functor*

$$\widetilde{w}_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{\Lambda_{-T}\cup\Lambda_T}\Big(\bigcup_{\alpha\in I} V_\alpha \times (0,\epsilon_\alpha)\Big),$$

where $T_- \to T_+ \to \widetilde{w}_\Lambda \circ m_\Lambda$ is an exact triangle.

Moreover, using the local-to-global argument, we also get an adjunction property following from the exact triangle [9, Section 4.5]. In summary, we have the following proposition.

Proposition 2.4. *Let $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be a proper subanalytic Legendrian subset. Let $\{U_\alpha\}_{\alpha\in I}$, $\{V_\alpha\}_{\alpha\in I}$ and $\{\epsilon_\alpha\}_{\alpha\in I}$ be a collection as above. Then the doubling functor is the left adjoint of microlocalization*

$$m^*_{\Lambda\times\mathbb{R}_{>0}} = \iota^*_{\Lambda\times\mathbb{R}_{>0}} \circ \widetilde{w}_\Lambda[-1] : \mu Sh_\Lambda(\Lambda) \to Sh_{\Lambda\times\mathbb{R}_{>0}\cup\Lambda_T}\Big(\bigcup_{\alpha\in I} V_\alpha \times (0,\epsilon_\alpha)\Big),$$

where $\iota^*_{\Lambda\times\mathbb{R}_{>0}}$ is the left adjoint to $\iota_{\Lambda\times\mathbb{R}_{>0}} : Sh_{\Lambda\times\mathbb{R}_{>0}\cup\Lambda_T}(M\times\mathbb{R}\times\mathbb{R}_{>0}) \hookrightarrow Sh(M\times\mathbb{R}\times\mathbb{R}_{>0})$.

When $U \cap \pi(\Lambda) = \varnothing$, note that we can choose $\mathscr{F}_U = 0_U$ and $\epsilon = +\infty$. When $\Lambda \subset T^{*,\infty}_{\tau>0}(M\times\mathbb{R})$ is compact, we can choose an open covering such that only finitely many open subsets intersect $\pi(\Lambda) \subset M \times \mathbb{R}$. Then there exists a uniform positive number

$$0 < \epsilon < \min_{\alpha\in I} \epsilon_\alpha = \min_{\alpha\in I,\, U_\alpha\cap\Lambda\neq\varnothing} \epsilon_\alpha.$$

In other words, we have

$$M \times \mathbb{R} \times \{\epsilon\} \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha)$$

Hence by restricting to $M \times \mathbb{R} \times \{\epsilon\}$, we get the doubling functor.

In general, there may not exist any $\epsilon > 0$ such that

$$M \times \mathbb{R} \times \{\epsilon\} \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha).$$

Hence it is difficult to construct the doubling functor for a uniform Reeb pushoff $\epsilon > 0$.

While there may not be a uniform $\epsilon > 0$ so that $M \times \mathbb{R} \times \{\epsilon\} \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha)$, there always exists some smooth function $\rho : M \times \mathbb{R} \to (0, +\infty)$ such that

$$\mathrm{Graph}(\rho) \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha).$$

Instead of restricting to the slice $M \times \mathbb{R} \times \{\epsilon\}$, we will restrict to the slice $\mathrm{Graph}(\rho)$.

Let the contact Hamiltonian be $\rho(x,t)$. The induced Hamiltonian vector field is $X_\rho = \rho(x,t)\partial_t - (\partial_x \rho(x,t) + \xi\, \partial_t \rho(x,t))\partial_\xi$, and the Hamiltonian diffeomorphism is

$$T_\rho(x, t; \xi) = (x, t + \rho(x,t); \xi - \partial_x \rho(x,t) - \xi\, \partial_t \rho(x,t)).$$

We will show that by restricting the doubled sheaf to the slice $\mathrm{Graph}(\rho)$, we get a sheaf with singular support on $T_{-\rho}(\Lambda) \cup T_\rho(\Lambda)$.

Proposition 2.5. *Let $\Lambda \subset J^1M$ be a proper subsanalytic Legendrian subset. Then for any C^1-small smooth function $\rho : M \times \mathbb{R} \to (0, +\infty)$, there exists a fully faithful functor*

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_{-\rho}(\Lambda) \cup T_\rho(\Lambda)}(M \times \mathbb{R}).$$

Proof. Since $\rho : M \times \mathbb{R} \to (0, +\infty)$ is C^1-small, we know that $\Lambda_{-T} \cup \Lambda_T$ is non-characteristic with respect to $\mathrm{Graph}(\rho) \subseteq M \times \mathbb{R} \times (0, +\infty)$. Hence by restricting $\widetilde{w}_\Lambda(\mathscr{F})$ to $\mathrm{Graph}(\rho)$, the singular support is contained in $T_{-\rho}(\Lambda) \cup T_\rho(\Lambda)$. Full faithfulness follows from the fact that up to the reparametrization

$$\phi_\rho : M \times \mathbb{R} \times (0, +\infty) \to M \times \mathbb{R} \times (0, +\infty),\ (x, t, \epsilon\rho(x,t)) \mapsto (x, t, \epsilon),$$

$\Lambda_{-T} \cup \Lambda_T$ is the Legendrian movie of a Hamiltonian flow. □

As explained at the beginning, for a general noncompact Legendrian, this is in fact the best we can do. In the following section, we will explain how to strengthen the result with the presence of a tubular neighbourhood of positive radius.

Proof of Theorem 2.1. Consider the tubular neighbourhood $J^1_{<\epsilon_0}\Lambda$ of radius $\epsilon_0 > 0$ of the noncompact Legendrian $\Lambda \subseteq J^1M$. We claim that there exists a contact Hamiltonian isotopy φ_H^1, $0 \leq t \leq 1$, such that

$$\varphi_H^1(T_{\pm\rho}(\Lambda)) = T_{\pm\epsilon}(\Lambda).$$

Consider the standard coordinates in $J^1_{<\epsilon_0}\Lambda \subseteq J^1\Lambda$. There exists a Legendrian isotopy from $T_{\pm\rho}(\Lambda)$ to $T_{\pm\epsilon}(\Lambda)$

$$\Lambda_{\pm u} = \{(x, (1-u)d\rho(x,t), ut + (1-u)\rho(x,t)) \mid (x,t) \in \Lambda \times \mathbb{R}\}.$$

Fix $\epsilon < \epsilon' < \epsilon_0$. Define a Hamiltonian function $H : \Lambda \times \mathbb{R} \to \mathbb{R}$ such that

$$H|_{\bigcup_{u \in [0,1]} \pi(T_{\pm u}(\Lambda))} = \pm(\epsilon - \rho(x,t)).$$

Then the corresponding contact Hamiltonian vector field

$$X_H = H\frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i}$$

is tangent to the fibers/leaves $T_x^*\Lambda \times \mathbb{R}$. Since the vector field is bounded on all the fibers, the integral flow is well defined for all time $u \geq 0$ and $\varphi_H^u(T_{\pm\rho}(\Lambda)) = T_{\pm u\epsilon \pm (1-u)\rho}(\Lambda)$. Then we can cut off the Hamiltonian such that

$$H|_{\bigcup_{u\in[0,1]}\Lambda_{\pm u}} = \pm(\epsilon - \rho(x,t)),\ H|_{J^1\Lambda\backslash J^1_{<\epsilon'}\Lambda} \equiv 0.$$

Therefore, the integral flow is supported in $J^1_{<\epsilon'}\Lambda$ and preserves the fibers/leaves. Hence the flow is well defined for all time $s \geq 0$.

The compact support condition then allows us to extend the Hamiltonian flow trivially from the tubular neighbourhood $J^1_{<\epsilon'}\Lambda$ to the ambient manifold J^1M. Using Guillermou-Kashiwara-Schapira [5], we can then conclude that there is an equivalence of sheaf categories

$$Sh_{T_{-\rho}(\Lambda)\cup T_\rho(\Lambda)}(M\times\mathbb{R}) \xrightarrow{\sim} Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M\times\mathbb{R}).$$

Then the result follows immediately from Proposition 2.5. □

There are definitely examples of noncompact Legendrians that no not admit a tubular neighbourhood of a positive radius with respect to any complete adapted metric. Conical Legendrian cobordisms are one class of such examples.

2.2. **Separation of double copies of the Legendrian.** Given the doubling on $T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)$ for some $\epsilon > 0$, to separate the double copies of the Legendrian, we need to apply some Hamiltonian isotopy from $\Lambda \cup T_\epsilon(\Lambda)$ to $\Lambda \cup T_u(\Lambda)$ for any $u > 0$. We show that this can again be done once there exists some tubular neighbourhood of positive radius.

Proposition 2.6. *Let $\Lambda \subset J^1M$ be any proper smooth Legendrian. When there exists a complete adapted metric on J^1M such that Λ admits a neighbourhood of positive radius $\epsilon/2 > 0$ disjoint from $\bigcup_{u\geq\epsilon} T_u(\Lambda)$, then for any $u > \epsilon$, there exists an equivalence induced by non-negative contact isotopies*

$$Sh_{\Lambda\cup T_\epsilon(\Lambda)}(M\times\mathbb{R}) \xrightarrow{\sim} Sh_{\Lambda\cup T_u(\Lambda)}(M\times\mathbb{R}).$$

Proof. Denote the Weinstein tubular neighbourhood of Λ by $U_\epsilon(\Lambda)$. By the assumption, we know that $U_{\epsilon/2}(\Lambda)$ and $\bigcup_{u\geq\epsilon} T_u(\Lambda)$ have a positive distance. Choose a cut-off function $H : J^1M \to \mathbb{R}$ such that

$$H|_{U_{\epsilon/2}(\Lambda)} \equiv 0,\ H|_{J^1M\backslash U_\epsilon(\Lambda)} \equiv 1,\ |dH| \leq 3/\epsilon < +\infty.$$

Since the metric on J^1M is adapted, we know that the Hamiltonian vector field $|X_H| \leq (1+9/\epsilon^2)^{1/2} < +\infty$. Since in addition that the metric on J^1M is complete, we know that the Hamiltonian flow φ_H^s exists for any $s \in \mathbb{R}$. Moreover, when $u \geq 0$,

$$\varphi_H^u\big(\Lambda \cup T_{2\epsilon/3}(\Lambda)\big) = \widetilde{L} \cup T_{\epsilon+u}(\widetilde{L}).$$

Therefore, by Guillermou-Kashiwara-Schapira [5], we can conclude that there is a canonical equivalence as in the statement of the proposition. □

Corollary 2.7. *Let $\Lambda \subset J^1M$ be any proper smooth Legendrian. When there exists a complete adapted metric on J^1M such that Λ admits a tubular neighbourhood of positive radius $\epsilon > 0$ disjoint from $\bigcup_{s\geq\epsilon} T_s(\Lambda)$, then for any $s > \epsilon$, there exists a fully faithful functor*

$$w_\Lambda :\ \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{\Lambda\cup T_s(\Lambda)}(M\times\mathbb{R}).$$

In particular, when the Lagrangian projection $\pi_{Lag}(\Lambda) \subset T^*M$ admits a tubular neighbourhood of positive radius, we always get the above functor.

Using the above results, we can in fact prove the sheaf quantization theorem for certain noncompact embedded Lagrangians that admit a tubular neighbourhood of a positive radius for some adapted metric.

Proof of Theorem 1.2. Let $\Lambda \subset J^1M$ be a proper Legendrian lift of the exact Lagrangian $L \subset T^*M$. Since the Lagrangian projection $L \subset T^*M$ admits a tubular neighbourhood of positive radius, by Corollary 2.7, for any $s > 0$, there exists a fully faithful functor

$$w_{\Lambda,s} : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{\Lambda \cup T_s(\Lambda)}(M \times \mathbb{R}).$$

Moreover, since for any $s < s'$, the contact isotopy from $\Lambda \cup T_s(\Lambda)$ to $\Lambda \cup T_{s'}(\Lambda)$ is non-negative by Proposition 2.6, by Guillermou–Kashiwara–Schapira [5, Proposition 4.4] or Kuo [8, Proposition 3.2], we have canonical continuation maps $w_{\Lambda,s}(\mathscr{F}) \to w_{\Lambda,s'}(\mathscr{F})$. Since the contact isotopy is cofinal by [2, Lemma 3.29], we know that it computes the colimit by positive wrapping away from Λ. Such colimit computes the left adjoint of the inclusion $\iota_\Lambda : Sh_\Lambda(M \times \mathbb{R}) \hookrightarrow Sh(M \times \mathbb{R})$ by [8, Theorem 1.2]:

$$\iota_\Lambda^*(\mathscr{F}) = \operatorname*{colim}_{s \to \infty} w_{\Lambda,s}(\mathscr{F}).$$

Full faithfulness of the wrapping functor follows from [8, Theorem 5.15] (technically, one needs to compactify $M \times \mathbb{R}$ to $M \times [-\infty, +\infty]$ and then apply the full faithfulness result, but the arguments are essentially the same). Alternatively, it also follows from the local constancy induced by continuation maps

$$\begin{aligned} Hom(w_{\Lambda,s}(\mathscr{F}), w_{\Lambda,s}(\mathscr{F})) &\xrightarrow{\sim} Hom(w_{\Lambda,s}(\mathscr{F}), w_{\Lambda,s'}(\mathscr{F})) \xrightarrow{\sim} Hom\Big(w_{\Lambda,s}(\mathscr{F}), \operatorname*{colim}_{s \to \infty} w_{\Lambda,s}(\mathscr{F})\Big) \\ &\xrightarrow{\sim} Hom(w_{\Lambda,s}(\mathscr{F}), \iota_\Lambda^*(\mathscr{F})) \xrightarrow{\sim} Hom(\iota_\Lambda^*(\mathscr{F}), \iota_\Lambda^*(\mathscr{F})) \end{aligned}$$

by [13, Lemma 2.10] or [8, Proposition 3.18]. □

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